

Mass Terms in Effective Theories of High Density Quark Matter

T. Schäfer^{1,2}

¹*Department of Physics, SUNY Stony Brook, Stony Brook, NY 11794*

²*Riken-BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973*

Abstract

We study the structure of mass terms in the effective theory for quasi-particles in QCD at high baryon density. To next-to-leading order in the $1/p_F$ expansion we find two types of mass terms, chirality conserving two-fermion operators and chirality violating four-fermion operators. In the effective chiral theory for Goldstone modes in the color-flavor-locked (CFL) phase the former terms correspond to effective chemical potentials, while the latter lead to Lorentz invariant mass terms. We compute the masses of Goldstone bosons in the CFL phase, confirming earlier results by Son and Stephanov as well as Bedaque and Schäfer. We show that to leading order in the coupling constant g there is no anti-particle gap contribution to the mass of Goldstone modes, and that our results are independent of the choice of gauge.

I. INTRODUCTION

Quark matter at high baryon density exhibits a rich phase structure [1–6], reminiscent of the many phases encountered in ordinary condensed matter systems. A particularly symmetric phase is the color-flavor-locked (CFL) phase of three flavor quark matter [7]. This phase is believed to be the true ground state of ordinary matter at very large density [8–10]. There are also speculations that CFL matter may exist in the core of neutron stars. At less-than-asymptotic densities relevant to astrophysical objects distortions of the pure CFL state due to non-zero quark masses are likely to be important [11–17]. In the present work we wish to study this problem through the use of two effective field theories.

The first of these, CFL chiral theory [18], is the chiral effective theory that governs the behavior of collective states with excitation energies smaller than the gap. Understanding the structure of mass terms in this theory allows us to determine the true ground state [15,17] and the spectrum of pseudo-Goldstone bosons [19–25] at finite quark mass. The second effective theory, high density effective theory [26,27], governs the interaction of quasi-particles and holes with excitation energies less than the Fermi energy in QCD at high baryon density. This theory simplifies perturbative QCD calculations in the limit where the Fermi momentum is very large, $p_F \rightarrow \infty$. Understanding the structure of mass corrections in this theory allows us to perform efficient calculations of mass terms in the CFL chiral theory. These calculations are based on the idea of matching. Matching expresses the requirement that Greens functions in the effective theories below and above the scale set by gap agree. In the present work, we will focus on a particularly simple quantity, the mass dependence of the vacuum energy.

This paper is organized as follows. In section II we introduce the effective theory for quasi-particles and holes in high density QCD and in section III we study the mass terms in this theory. In sections IV and V we study the possible role of anti-particle gap parameters. In section VI we discuss the matching calculation that determines the masses of Goldstone bosons in the CFL chiral theory.

II. HIGH DENSITY EFFECTIVE THEORY (HDET)

The QCD Lagrangian in the presence of a chemical potential is given by

$$\mathcal{L} = \bar{\psi} (i\mathcal{D} + \mu\gamma_0) \psi - \bar{\psi}_L M \psi_R - \bar{\psi}_R M^\dagger \psi_L - \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a, \quad (1)$$

where M is a complex quark mass matrix which transforms as $M \rightarrow LMR^\dagger$ under chiral transformations $(L, R) \in SU(3)_L \times SU(3)_R$ and μ is the baryon chemical potential. In the vicinity of the Fermi surface the relevant degrees of freedom are particle and hole excitations which move with the Fermi velocity v_F . We shall describe these excitations in terms of the field $\psi_+(\vec{v}_F, x)$. At tree level, the quark field ψ can be decomposed as $\psi = \psi_+ + \psi_-$ where $\psi_\pm = \frac{1}{2}(1 \pm \vec{\alpha} \cdot \hat{v}_F)\psi$. Integrating out the ψ_- field at leading order in $1/p_F$ we get [26–28,24]

$$\begin{aligned} \mathcal{L} = & \psi_{L+}^\dagger (iv \cdot D) \psi_{L+} - \frac{\Delta}{2} \left(\psi_{L+}^{ai} C \psi_{L+}^{bj} (\delta_{ai} \delta_{bj} - \delta_{aj} \delta_{bi}) + \text{h.c.} \right) \\ & - \frac{1}{2p_F} \psi_{L+}^\dagger \left((\mathcal{D}_\perp)^2 + MM^\dagger \right) \psi_{L+} + (R \leftrightarrow L, M \leftrightarrow M^\dagger) + \dots, \end{aligned} \quad (2)$$

where $D_\mu = \partial_\mu + igA_\mu$, $v_\mu = (1, \vec{v})$ and i, j, \dots and a, b, \dots denote flavor and color indices. The longitudinal and transverse components of a vector B_μ are defined by $(B_0, \vec{B})_\parallel = (B_0, \vec{v}(\vec{B} \cdot \vec{v}))$ and $(B_\mu)_\perp = B_\mu - (B_\mu)_\parallel$. In order to perform perturbative calculations in the superconducting phase we have added a tree level gap term $\psi_{L,R} C \Delta \psi_{L,R}$. For consistency we have to subtract this term from the interacting part of the Lagrangian, $\mathcal{L}_{int} = -\psi_{L,R} C \Delta \psi_{L,R}$. The magnitude of Δ is determined order by order in perturbation theory from the requirement that the free energy is stationary with respect to Δ .

III. MASS CONTRIBUTION TO THE VACUUM ENERGY

We would like to compute the shift in vacuum energy due to non-zero quark masses and match the result against the shift computed in the effective theory for the Goldstone bosons. For this purpose we need to determine mass corrections to the high density effective theory. At $O(1/p_F)$ there is only one operator,

$$\mathcal{L} = -\frac{1}{2p_F} \left(\psi_{L+}^\dagger M M^\dagger \psi_{L+} + \psi_{R+}^\dagger M^\dagger M \psi_{R+} \right). \quad (3)$$

This term arises from expanding the kinetic energy of a massive fermion around $p = p_F$. At $O(1/p_F)$ there is no mass correction to the quark-gluon vertex. Indeed, integrating out the ψ_- field we get

$$\mathcal{L} = \frac{g}{2p_F} \psi_{+,L}^\dagger \left[\gamma_0 M \vec{\alpha} \cdot \vec{A}_\perp + \vec{\alpha} \cdot \vec{A}_\perp \gamma_0 M \right] \psi_{+,R} = 0. \quad (4)$$

There is also no mass correction to the emission of an electric gluon. This is the case because $\psi_-^\dagger(\vec{v}') g A_0 \psi_+(\vec{v})$ vanishes in the forward direction, $\vec{v}' = \vec{v}$. There is, however, a mass dependent four-fermion contact term induced by electric gluon exchange. One way to see this is to consider off-forward amplitudes with $\vec{v}' \neq \vec{v}$, integrate out hard gluon exchanges, and take the forward limit $\vec{v}' \rightarrow \vec{v}$ in the end. In the case $\vec{v}' \neq \vec{v}$ we find, after integrating out ψ_- at tree level, a mass correction to the electric gluon vertex

$$\mathcal{L} = -\frac{g}{p_F} \psi_{L,+}^\dagger(\vec{v}') \gamma_0 M A_0 \psi_{R,+}(\vec{v}) + (L \leftrightarrow R, M \leftrightarrow M^\dagger). \quad (5)$$

This term still vanishes in the limit $\vec{v}' \rightarrow \vec{v}$, but it gives a non-vanishing contribution to quark-quark scattering in the forward direction. This is the case because the vanishing amplitude in the forward direction cancels against the collinear singularity in the gluon propagator.

We can construct the four-fermion operator by integrating out hard gluons with momenta $p \sim p_F$. Because the vertex (5) vanishes in the forward direction the contribution from soft gluons with $p \sim gp_F$ is suppressed. We find

$$\begin{aligned} \mathcal{L} = & \frac{g^2}{8p_F^2} \left[\left(\psi_{+,L}^\dagger(\vec{v}') \gamma_0 M \lambda^a \psi_{+,R}(\vec{v}) \right) \left(\psi_{+,L}^\dagger(\vec{v}') \gamma_0 M \lambda^a \psi_{+,R}(\vec{v}) \right) \right. \\ & + \left(\psi_{+,L}^\dagger(\vec{v}') \gamma_0 M \lambda^a \psi_{+,R}(\vec{v}) \right) \left(\psi_{+,R}^\dagger(\vec{v}) \gamma_0 M^\dagger \lambda^a \psi_{+,L}(\vec{v}') \right) + (L \leftrightarrow R, M \leftrightarrow M^\dagger) \Big] \\ & \cdot \frac{1}{2p_F^2(1 - \vec{v}' \cdot \vec{v})} \end{aligned} \quad (6)$$

We can make the $\vec{v}' \rightarrow \vec{v}$ limit more explicit by Fierz-rearranging the four-fermion terms in (6). For the first term we use

$$\begin{aligned}
& \left(\psi_L^\dagger(\vec{v}') \gamma_0 \psi_R(\vec{v}) \right) \left(\psi_L^\dagger(\vec{v}') \gamma_0 \psi_R(\vec{v}) \right) \\
&= \frac{1}{2} \left[\left(\psi_L^\dagger(\vec{v}') C \psi_L^\dagger(\vec{v}') \right) \left(\psi_R(\vec{v}) C \psi_R(\vec{v}) \right) - \left(\psi_L^\dagger(\vec{v}') \bar{\alpha} C \psi_L^\dagger(\vec{v}') \right) \left(\psi_R(\vec{v}) C \bar{\alpha} \psi_R(\vec{v}) \right) \right] \\
&= \frac{1}{2} \left(\psi_L^\dagger C \psi_L^\dagger \right) \left(\psi_R C \psi_R \right) (1 - \vec{v} \cdot \vec{v}').
\end{aligned} \tag{7}$$

Here we have dropped the subscript “+” and suppressed the color and flavor structure of the fields. A similar identity can be derived for the second term in (6). We observe that the factor $1 - \vec{v} \cdot \vec{v}'$ cancels and that in the limit $\vec{v}' \rightarrow \vec{v}$ we are left with a local four-fermion interaction. Collecting (3) and (6) we get the following result for the mass terms in the high density effective theory to $O(1/p_F^4)$

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{2p_F} \left(\psi_L^\dagger M M^\dagger \psi_L + \psi_R^\dagger M^\dagger M \psi_R \right) \\
& + \frac{g^2}{8p_F^4} \left((\psi_L^{A\dagger} C \psi_L^{B\dagger}) (\psi_R^C C \psi_R^D) \Gamma^{ABCD} + (\psi_L^{A\dagger} \psi_L^B) (\psi_R^{C\dagger} \psi_R^D) \tilde{\Gamma}^{ACBD} \right) \\
& + (L \leftrightarrow R, M \leftrightarrow M^\dagger) + \dots
\end{aligned} \tag{8}$$

Here, we have rewritten the four-fermion terms using the CFL eigenstates ψ^A defined by $\psi_i^a = \psi^A (\lambda^A)_{ai} / \sqrt{2}$, $A = 0, \dots, 8$ where the Gell-Mann matrices λ^i are normalized as $\text{Tr}[\lambda^i \lambda^j] = 2\delta^{ij}$ and $\lambda^0 = \sqrt{2/3}$. This is simply a choice of basis, it does not imply that (8) is only valid in the CFL phase. The tensors Γ and $\tilde{\Gamma}$ are defined by

$$\begin{aligned}
\Gamma^{ABCD} &= \frac{1}{8} \left\{ \text{Tr} \left[\lambda^A M (\lambda^D)^T \lambda^B M (\lambda^C)^T \right] - \frac{1}{3} \text{Tr} \left[\lambda^A M (\lambda^D)^T \right] \text{Tr} \left[\lambda^B M (\lambda^C)^T \right] \right\}, \\
\tilde{\Gamma}^{ABCD} &= \frac{1}{8} \left\{ \text{Tr} \left[\lambda^A M (\lambda^D)^T \lambda^C M^\dagger (\lambda^B)^T \right] - \frac{1}{3} \text{Tr} \left[\lambda^A M (\lambda^D)^T \right] \text{Tr} \left[\lambda^C M^\dagger (\lambda^B)^T \right] \right\}.
\end{aligned} \tag{9}$$

We should note that equ. (8) can also be derived directly, by computing the chirality violating quark-quark scattering amplitude $T((q_R)_i^a + (q_R)_j^b \rightarrow (q_L)_k^c + (q_L)_l^d)$ as well as the mass correction to $T((q_R)_i^a + (q_L)_j^b \rightarrow (q_L)_k^c + (q_R)_l^d)$. The corresponding tree level diagrams are shown in Fig. 1. We find that to leading order in g the scattering amplitudes are independent of the scattering angle and that they can be represented by the contact terms in equ. (8). We also observe that the effective lagrangian (8) is consistent with the approximate gauge symmetry of Bedaque and Schäfer [15]. This approximate gauge symmetry reflects the fact

that to leading order in $1/p_F$ mass terms appear as effective chemical potentials $MM^\dagger/(2p_F)$ and $M^\dagger M/(2p_F)$ for left and right handed fermions, respectively.

We can now compute the shift in the vacuum energy due to the mass terms in equ. (8). At $O(M^2)$ and to leading order in $1/p_F$ there is a contribution of the form $\text{Tr}[MM^\dagger]$ from the two-fermion operators in equ. (8). This contribution is the same in the normal and superfluid phases, and it does not correspond to a Goldstone boson mass term. This is the case because $\text{Tr}[MM^\dagger]$ cannot be matched against a term in the CFL chiral theory which contains the chiral field Σ . At $O(M^2)$, the only terms in the vacuum energy which correspond to Goldstone boson mass terms are $\text{Tr}[M^2]$, $(\text{Tr}[M])^2$ and $\text{Tr}[M]\text{Tr}[M^\dagger]$.

At next-to-leading order in $1/p_F$ there is a mass correction to the vacuum energy due to the four-fermion operators in equ. (8). This contribution comes from the diagrams shown in Fig. 2. The first diagram is proportional to the square of the superfluid density

$$\langle \psi_L^A C \psi_L^B \rangle = \delta^{AB} \int \frac{d^4 p}{(2\pi)^4} \frac{\Delta_A(p_0)}{p^2 - \epsilon_p^2 - \Delta_A^2(p_0)} = \delta^{AB} \Delta^A \frac{3\sqrt{2}\pi}{g} \left(\frac{p_F^2}{2\pi^2} \right), \quad (10)$$

with $\Delta^A = C^A \Delta$ and $C^A = (2, -1)$ for $A = (0, 1 \dots 8)$. In deriving (10) we have used the approximate solution of the gap equation $\Delta(p_0) = \Delta(p_0 = 0) \sin(\frac{g}{3\sqrt{2}\pi} \log(\mu/p_0))$. The color-flavor factor is given by

$$\sum_{A,B=0}^8 C^A C^B \Gamma^{AABB} = -\frac{4}{3} \left\{ (\text{Tr}[M])^2 - \text{Tr}[M^2] \right\}. \quad (11)$$

The second diagram is proportional to $\text{Tr}[MM^\dagger]$ and does not contribute to Goldstone boson masses. We note that there is no contribution of the form $\text{Tr}[M]\text{Tr}[M^\dagger]$. Using eqns. (10) and (11) we find that the shift in the vacuum energy due to the first diagram in Fig. 2 is given by

$$\Delta\mathcal{E} = -\frac{3\Delta^2}{4\pi^2} \left\{ (\text{Tr}[M])^2 - \text{Tr}[M^2] \right\} + (M \leftrightarrow M^\dagger), \quad (12)$$

which agrees with the result of Son and Stephanov [19]. The two-fermion operators in equ. (8) contribute to the masses of Goldstone bosons at $O(M^4)$. The corresponding term in the vacuum energy was computed in [15]

$$\Delta\mathcal{E} = \frac{m_D^2}{8p_F^2} \text{Tr} \left[(MM^\dagger)(M^\dagger M) - (MM^\dagger)^2 \right], \quad (13)$$

where $m_D^2 = (21 - 8 \log(2))p_F^2/(36\pi^2)$ is (up to a factor of g^2) the Debye mass in the CFL phase. There are other contributions to the vacuum energy at $O(M^4)$, but these terms are suppressed by additional powers of $1/p_F$.

IV. ANTI-PARTICLE GAP CONTRIBUTION

Several authors have suggested that the calculation of the mass shift in the vacuum energy, and consequently the masses of Goldstone bosons, requires the knowledge of the “anti-gap”, that is the gap parameter for the ψ_- excitation [21–25]. As we shall see below, the anti-particle gap contribution to the vacuum energy is $\Delta\mathcal{E} \sim g^{-1}\Delta\bar{\Delta}M^2$. We shall also see that the natural magnitude of the anti-particle gap is $\bar{\Delta} \simeq \Delta$. In this case, the anti-particle gap contribution to $\Delta\mathcal{E}$ dominates over the results we obtained in the previous section. This in itself may not pose a problem, but we shall also see that a straightforward calculation of the anti-particle gap in the microscopic theory does not seem to produce a gauge invariant result [30].

In order to study the problem in more detail we consider the QCD lagrangian in the presence of gap parameters for both the ψ_+ and ψ_- excitations,

$$\begin{aligned} \mathcal{L} = & \psi_{+,L}^\dagger (i v \cdot D) \psi_{+,L} - \psi_{+,L}^\dagger (i \vec{\alpha}_\perp \cdot \vec{D}) \psi_{-,L} - \psi_{-,L}^\dagger (i \vec{\alpha}_\perp \cdot \vec{D}) \psi_{+,L} + \psi_{-,L}^\dagger (2p_F + i \tilde{v} \cdot D) \psi_{-,L} \quad (14) \\ & + \frac{1}{2} \left(\Delta^{AB} \psi_{+,L}^A C \psi_{+,L}^B + \tilde{\Delta}_1^{AB} \psi_{-,L}^A C \gamma_0 \psi_{+,R}^B + \tilde{\Delta}_2^{AB} \psi_{+,R}^A C \gamma_0 \psi_{-,L}^B \right. \\ & \quad \left. + \bar{\Delta}^{AB} \psi_{-,L}^A C \psi_{-,L}^B + (h.c.) \right) \\ & - \psi_{-,L}^\dagger \gamma_0 M \psi_{+,R} - \psi_{+,L}^\dagger \gamma_0 M \psi_{-,R} + (L \leftrightarrow R, M \leftrightarrow M^\dagger), \end{aligned}$$

where $\tilde{v} = (1, -\vec{v})$. Here, we have included all spin 0 gaps that can be constructed from ψ_\pm . In the CFL phase the particle gap has the structure $\Delta^{AB} = \mathbf{\Delta}^{AB} \Delta$ with $\mathbf{\Delta}^{AB} = \delta^{AB} C^A$ and $C^A = (2, -1)$ for $A = (0, 1 \dots 8)$. We shall see that the anti-particle gap has the same structure. We will determine the color-flavor structure of the mixed particle-anti-particle

gap $\tilde{\Delta}$ below. In the case of massless quarks, $M = 0$, the mixed gap $\tilde{\Delta}$ vanishes in the weak coupling approximation [1,34,35]. Since we are interested in mass corrections to the vacuum energy density, $\tilde{\Delta}$ cannot be neglected. It is straightforward to integrate out ψ_- at tree level. We find

$$\begin{aligned} \mathcal{L} = & \psi_{+,L}^\dagger (iv \cdot D) \psi_{+,L} - \frac{1}{2p_F} \psi_{+,L} (\not{D}_\perp)^2 \psi_{+,L} - \frac{1}{2p_F} \psi_{+,L}^\dagger (MM^\dagger) \psi_{+,L} \\ & + \frac{1}{2} \left(\Delta^{AB} \psi_{+,L}^A C \psi_{+,L}^B - \frac{1}{2p_F} (M^T \tilde{\Delta}_1 - \tilde{\Delta}_2 M)^{AB} \psi_{+,R}^A C \psi_{+,R}^B \right. \\ & \left. - \frac{1}{4p_F^2} (M^T \overline{\Delta} M)^{AB} \psi_{+,R}^A C \psi_{+,R}^B + (h.c.) \right) + (L \leftrightarrow R, M \leftrightarrow M^\dagger), \end{aligned} \quad (15)$$

where $M^{AB} = \frac{1}{2} \text{Tr}[\lambda^A M (\lambda^B)^T]$ is the quark mass matrix $\delta^{ab} M_{ij}$ in the CFL basis. We can now determine the contribution of $\overline{\Delta}$ and $\tilde{\Delta}$ to the mass terms in the vacuum energy. Computing the diagrams in Fig. 3 we find

$$\begin{aligned} \Delta \mathcal{E} = & -\frac{3\pi^2}{2\sqrt{2}g} \left(\frac{p_F^2}{2\pi^2} \right) \left\{ \frac{1}{(2p_F)^2} \text{Tr}(M^T \overline{\Delta} M \Delta) + \frac{1}{2p_F} \text{Tr}(M^T \tilde{\Delta}_1 \Delta - M \Delta \tilde{\Delta}_2) \right\} \\ & + (M \leftrightarrow M^\dagger). \end{aligned} \quad (16)$$

We can analyze the anti-particle gap contribution in more detail. Assuming that the anti-particle gap $\overline{\Delta}$ has the same color-flavor structure as the particle gap Δ we find

$$(M^T \overline{\Delta} M)^{AB} = -\frac{\overline{\Delta}}{2} \left\{ \text{Tr}(M \lambda^A M \lambda^B) - \text{Tr}(M \lambda^A) \text{Tr}(M \lambda^B) \right\}. \quad (17)$$

This allows us to write the anti-particle gap contribution to the vacuum energy as

$$\Delta \mathcal{E} = -\frac{3}{8\sqrt{2}g} \Delta \overline{\Delta} \left\{ (\text{Tr}[M])^2 - \text{Tr}[M^2] \right\}. \quad (18)$$

This result agrees, up to a factor of 1/2, with the result of Beane et al. [24]. The factor 1/2 discrepancy is due to the fact that Beane et al. compute loop integrals under the assumption that the gap is not a function of energy or momentum.

V. ANTI-PARTICLE GAP PARAMETERS

After we have determined the contribution of the anti-particle gap parameters $\overline{\Delta}$ and $\tilde{\Delta}$ to the mass terms in the vacuum energy we shall now try to compute the gap parameters

to leading order in perturbation theory. This calculation is performed in the microscopic theory, but we will use kinematic simplifications that are similar to the approximations that are built into the effective theory. In principle, computing the anti-particle gap parameters requires solving a coupled set of gap equations for Δ , $\overline{\Delta}$ and $\tilde{\Delta}$. However, both $\overline{\Delta}$ and $\tilde{\Delta}$ are induced gaps. This means that even though $\overline{\Delta}$ and $\tilde{\Delta}$ are not necessarily small, the correction to the particle gap Δ due to the fact that the anti-particle gaps are non-zero is suppressed by $1/p_F$. As a consequence, only the particle gap Δ has to be determined self-consistently. Both $\overline{\Delta}$ and $\tilde{\Delta}$ can be calculated perturbatively.

In the last section we saw that $\overline{\Delta}$ contributes to \mathcal{E} at $O(M^2)$ whereas $\tilde{\Delta}$ contributes at $O(M)$. However, $\tilde{\Delta}$ itself is of order $(M/p_F)\Delta$. Therefore, both gap parameters effectively contribute to \mathcal{E} at the same in order in M . In order to collect all contributions to the vacuum energy at $O(M^2)$, we shall compute Δ to $O(M^2)$, $\tilde{\Delta}$ to $O(M)$, and $\overline{\Delta}$ to $O(1)$. In practice this is most easily achieved by considering the Dyson-Schwinger equation [1]

$$\Sigma(k) = -ig^2 \int \frac{d^4q}{(2\pi)^4} \Gamma_\mu^a S(q) \Gamma_\nu^b D_{\mu\nu}^{ab}(q-k). \quad (19)$$

and expanding the quark propagator S in powers of M ,

$$S = S_0 + S_0 \mathcal{M} S_0 + S_0 \mathcal{M} S_0 \mathcal{M} S_0 + \dots \quad (20)$$

Here, $\Sigma(k)$ is the proper self energy in the Nambu-Gorkov formalism, Γ_μ^a is the quark-gluon vertex and $D_{\mu\nu}^{ab}(q-k)$ is the gluon propagator. To leading order in the perturbative expansion, we can use the free vertex

$$\Gamma_\mu^a = \begin{pmatrix} \gamma_\mu \frac{\lambda^a}{2} & 0 \\ 0 & -\gamma_\mu \left(\frac{\lambda^a}{2}\right)^T \end{pmatrix}. \quad (21)$$

To first order in $1/p_F$, the Nambu-Gorkov propagator S_0 is given by

$$S_0(q) = \begin{pmatrix} \frac{q_0 + \epsilon_q}{D_+(q)} \gamma_0 \Lambda_q^- + \frac{1}{2p_F} \gamma_0 \Lambda_q^+ & \frac{\Delta}{D_+(q)} \Lambda_q^+ \\ \frac{\Delta}{D_+(q)} \Lambda_q^- & \frac{q_0 - \epsilon_q}{D_+(q)} \gamma_0 \Lambda_q^+ - \frac{1}{2p_F} \gamma_0 \Lambda_q^- \end{pmatrix}. \quad (22)$$

Here, $\Lambda_q^\pm = \frac{1}{2}(1 \pm \vec{\alpha} \cdot \hat{q})$, $D_+(q) = (\epsilon_q^2 + \Delta^2)^{1/2}$ and $\epsilon_q = |\vec{q}| - p_F$. The quark mass matrix has the structure

$$\mathcal{M} = \begin{pmatrix} MP_R + M^\dagger P_L & 0 \\ 0 & M^T P_R + M^* P_L \end{pmatrix}, \quad (23)$$

where $P_{L,R}$ are left and right handed projection operators. In order to determine the gap parameters we only need the S_{21} component of the Nambu-Gorkov propagator. To $O(M^2)$ we find

$$\begin{aligned} S_{21}(q) = & \frac{\Delta}{D_+(q)}(P_R - P_L)\Lambda_q^- \\ & - \frac{1}{2p_F D_+(q)}\gamma_0 \left(M^T \Delta P_R \Lambda_q^- - \Delta M P_L \Lambda_q^+ \right) \\ & - \frac{1}{4p_F^2 D_+(q)} M^T \Delta M P_L \Lambda_q^+ + \frac{q_0 + \epsilon_q}{2p_F D_+(q)^2} \Delta M M^\dagger P_R \Lambda_q^- + \frac{q_0 - \epsilon_q}{2p_F D_+(q)^2} M^T M^* \Delta P_L \Lambda_q^- \\ & + (M \leftrightarrow M^\dagger, L \leftrightarrow R) + \dots \end{aligned} \quad (24)$$

In order to solve the gap equation (19) we also have to specify the gluon propagator. In a general covariant gauge the gluon propagator is given by

$$D_{\mu\nu}(q) = \frac{P_{\mu\nu}^T}{q^2 - G} + \frac{P_{\mu\nu}^L}{q^2 - F} - \xi \frac{q_\mu q_\nu}{q^4} \quad (25)$$

where the self energies D and F are functions of q_0 and $|\vec{q}|$, ξ is a gauge parameter, and the projectors $P_{\mu\nu}^{T,L}$ are defined by

$$P_{ij}^T = \delta_{ij} - \hat{q}_i \hat{q}_j, \quad P_{00}^T = P_{0i}^T = 0, \quad (26)$$

$$P_{\mu\nu}^L = -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} - P_{\mu\nu}^T. \quad (27)$$

To leading order in the coupling constant g , and in the limit $q_0 < |\vec{q}| \ll p_F$ the electric and magnetic self energies F, G are given by

$$F = 2m^2, \quad G = \frac{i\pi}{2} m^2 \frac{q_0}{|\vec{q}|}, \quad (28)$$

where $m^2 = N_f g^2 \mu^2 / (4\pi^2)$. We can now compute the gap parameters using the methods explained in [30]. If we only take into account the first term in the anomalous quark propagator equ. (24) we get

$$\Delta^{AB} = \Delta^{AB} \frac{g^2}{12\pi^2} \int dq_0 \int dx \left(\frac{\frac{3}{2} - \frac{1}{2}x}{1 - x + G/(2p_F^2)} + \frac{\frac{1}{2} + \frac{1}{2}x}{1 - x + F/(2p_F^2)} \right) \frac{\Delta(q_0)}{\sqrt{q_0^2 + \Delta(q_0)^2}}. \quad (29)$$

Here, $\Delta^{AB} = \delta^{AB} C^A$ with $C^A = (2, -1)$ for $A = (0, 1 \dots 8)$ is a constant matrix with the color-flavor structure of the CFL condensate. $\Delta(q_0)$ is the absolute magnitude of the CFL gap on the quasi-particle mass shell. To leading order the equation (29) can be solved by taking $x \simeq 1$ in the numerator. This means that we approximate scattering amplitudes by their values in the forward direction. The particle gap on the Fermi surface $\Delta \equiv \Delta(p_0 = 0)$ is given by [29–33]

$$\Delta = 512\pi^4 (2/3)^{-5/2} b'_0 \mu g^{-5} \exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right), \quad (30)$$

where b'_0 is a constant which is determined by non-Fermi liquid effects [33] that are not included in our calculation. In the same fashion we can also compute the gap for anti-particles. Again, we only need the first term in the anomalous quark propagator equ. (24).

We find

$$\begin{aligned} \overline{\Delta}^{AB} = \Delta^{AB} \frac{g^2}{12\pi^2} \int dq_0 \int dx \left(\frac{\frac{1}{2} + \frac{1}{2}x}{1 - x + G/(2p_F^2)} + \frac{\frac{1}{2} - \frac{1}{2}x}{1 - x + F/(2p_F^2)} \right. \\ \left. + \frac{\xi}{1 - x + q_0^2/(2p_F^2)} \right) \frac{\Delta(q_0)}{\sqrt{q_0^2 + \Delta(q_0)^2}}. \end{aligned} \quad (31)$$

We observe that the anti-particle gap has the same color-flavor structure as the particle gap. To leading order, we can again approximate $x \simeq 1$ in the numerator. We note that in this limit there is no contribution from electric gluons. We also find that the gauge parameter ξ does not disappear from (31). We obtain

$$\overline{\Delta}^{AB} = \Delta^{AB} (1 + 3\xi) + O(g\Delta). \quad (32)$$

The next step is to include linear mass terms in the propagator (24). These terms mix left and right handed fermions and contribute to the mixed particle-anti-particle gaps $\tilde{\Delta}_{1,2}$. We find

$$\begin{aligned} (\tilde{\Delta}_1)^{AB} = -(\Delta M)^{AB} \frac{g^2}{12\pi^2 (2p_F)} \int dq_0 \int dx \left(\frac{\frac{1}{2} + \frac{1}{2}x}{1 - x + G/(2p_F^2)} - \frac{\frac{1}{2} - \frac{1}{2}x}{1 - x + F/(2p_F^2)} \right. \\ \left. + \frac{\xi}{1 - x + q_0^2/(2p_F^2)} \right) \frac{\Delta(q_0)}{\sqrt{q_0^2 + \Delta(q_0)^2}}, \end{aligned} \quad (33)$$

as well as the analogous equation for $\tilde{\Delta}_2$ with $(\Delta M) \rightarrow -(M^T \Delta)$. In equ. (33) we dropped contributions that are proportional to M^\dagger as they give terms in the vacuum energy that are proportional to $\text{Tr}(MM^\dagger)$. We note that the magnetic contribution and the gauge parameter term have the same structure as the corresponding terms in the equation for the anti-particle gap. The only difference is that the relative sign of the contribution from electric gluons is different. We get

$$(\tilde{\Delta}_1)^{AB} = -(\Delta M)^{AB} \frac{1+3\xi}{2p_F} + O(g\Delta), \quad (34)$$

$$(\tilde{\Delta}_2)^{AB} = +(M^T \Delta)^{AB} \frac{1+3\xi}{2p_F} + O(g\Delta). \quad (35)$$

Finally we determine the $O(M^2)$ shift in the particle gap. Again, we drop the MM^\dagger terms in the anomalous quark propagator equ. (24). If we write the particle gap as $\Delta^{AB} = \Delta_0^{AB} + \Delta_2^{AB} + \dots$ where Δ_0^{AB} is the gap in the limit $M \rightarrow 0$ and Δ_2^{AB} is the $O(M^2)$ shift we obtain

$$(\Delta_2)^{AB} = -(M^T \Delta M)^{AB} \frac{g^2}{12\pi^2(2p_F)^2} \int dq_0 \int dx \left(\frac{\frac{1}{2} + \frac{1}{2}x}{1-x+G/(2p_F^2)} + \frac{\frac{1}{2} - \frac{1}{2}x}{1-x+F/(2p_F^2)} + \frac{\xi}{1-x+q_0^2/(2p_F^2)} \right) \frac{\Delta(q_0)}{\sqrt{q_0^2 + \Delta(q_0)^2}}, \quad (36)$$

We note that again, the structure of the right hand side is the same as in the equation for the anti-particle gap. We find

$$(\Delta_2)^{AB} = -(M^T \Delta M)^{AB} \frac{1+3\xi}{(2p_F)^2} + O(g\Delta). \quad (37)$$

The gap equations for the anti-particle gaps are schematically shown in Fig. 4. We can now collect all $O(M^2)$ contributions to the vacuum energy density. Using (16) and including the $O(M^2)$ shift in the gap we have

$$\Delta\mathcal{E} = -\frac{3\pi^2}{2\sqrt{2}g} \left(\frac{p_F^2}{2\pi^2} \right) \left\{ -\text{Tr}(\Delta_2 \Delta) + \frac{1}{2p_F} \text{Tr}(M^T \tilde{\Delta}_1 \Delta - M \Delta \tilde{\Delta}_2) + \frac{1}{(2p_F)^2} \text{Tr}(M^T \overline{\Delta} M \Delta) \right\} + (M \leftrightarrow M^\dagger). \quad (38)$$

Using the results obtained in this section, we can express the vacuum energy in terms of the particle gap only. We find

$$\Delta\mathcal{E} = -\frac{3}{16\sqrt{2}g}(1+3\xi)\left\{\text{Tr}\left(M^T\Delta M\Delta\right) - 2\text{Tr}\left(M^T\Delta M\Delta\right) + \text{Tr}\left(M^T\Delta M\Delta\right)\right\} = 0. \quad (39)$$

We observe that, to leading order in g , the net contribution to the vacuum energy from $\overline{\Delta}$, $\tilde{\Delta}_{1,2}$ and Δ_2 vanishes. Individually, all these terms are non-zero but they depend on the gauge parameter and therefore have no physical meaning. We also note that the leading order results for the anti-particle gap parameters only involve magnetic gluon exchanges. The cancellation that we obtained in this section is automatically taken into account in the effective theory of section III because of the relation (4). Our results indicate that it is not necessary to explicitly include anti-particle gaps in the high density effective theory. There is no instability in the anti-gap channel, and therefore no reason to include anti-gaps as variational parameters. The loop diagrams which determine the anti-particle gaps are dominated by quasi-particles in the vicinity of the Fermi surface. This means that there is also no reason to include anti-gaps as effective operators governed by short distance effects in the microscopic theory. The anti-gap contribution to the vacuum energy is automatically included in the effective theory discussed in section III.

Indeed, at next-to-leading order in g there is a non-vanishing term in the vacuum energy which arises from the contribution of electric gluons to the anti-particle gaps. This can be seen from the fact that the electric gluon contribution appears with a different sign in eqns. (31,36) as compared to equ. (33). This means that the cancellation observed in the magnetic sector does not occur in the electric sector. It is precisely this effect that we computed in section III using an effective four-fermion vertex. The fact that this is possible can be seen from the structure of the gap equations. To leading order in g , both the numerator and the denominator of the electric gluon contribution in eqns. (31,33,36) are proportional to $(1-x)$. As a result, the electric gluon contribution is effectively point-like.

VI. CFL CHIRAL THEORY (CFL χ TH)

In this section we shall discuss how to match the vacuum energy calculated in the high density effective theory to the vacuum energy in the effective chiral theory for the Goldstone

modes in the color-flavor-locked phase. Since all the relevant steps have been discussed in the literature, we can be very brief in our presentation. The leading terms in the effective chiral Lagrangian take the form

$$\begin{aligned} \mathcal{L}_{eff} = & \frac{f_\pi^2}{4} \text{Tr} \left[\nabla_0 \Sigma \nabla_0 \Sigma^\dagger - v_\pi^2 \partial_i \Sigma \partial_i \Sigma^\dagger \right] + \frac{3f_{\eta'}^2}{4} \left[\partial_0 V \partial_0 V^* - v_{\eta'}^2 \partial_i V \partial_i V^* \right] \\ & + \left[A_1 \text{Tr}(M \Sigma^\dagger) \text{Tr}(M \Sigma^\dagger) V + A_2 \text{Tr}(M \Sigma^\dagger M \Sigma^\dagger) V + A_3 \text{Tr}(M \Sigma^\dagger) \text{Tr}(M^\dagger \Sigma) + h.c. \right] + \dots \end{aligned} \quad (40)$$

Here $\Sigma = \exp(i\phi^a \lambda^a / f_\pi)$ is the octet field, f_π is the pion decay constant, and $A_{1,2,3}$ are the coefficients of the $O(M^2)$ mass terms. We have not displayed any field independent MM^\dagger terms. The axial $U(1)_A$ field is $V = \exp(4i\theta) = \exp(2i\eta' / (\sqrt{6}f_{\eta'}))$. As explained in [15] the chirality conserving two-fermion operators (3) act like an effective chemical potential. In the effective chiral theory they appear as gauge fields,

$$\nabla_0 \Sigma = \partial_0 \Sigma + i \left(\frac{MM^\dagger}{2p_F} \right) \Sigma - i \Sigma \left(\frac{M^\dagger M}{2p_F} \right). \quad (41)$$

The coefficients $A_{1,2,3}$ have to be matched against the $O(M^2)$ result for the vacuum energy (12). We find

$$A_1 = -A_2 = \frac{3\Delta^2}{4\pi^2}, \quad A_3 = 0, \quad (42)$$

which agrees with the result of Son and Stephanov. We can now compute the masses of Goldstone bosons in the CFL phase. Bedaque and Schäfer argued that the expansion parameter in the chiral expansion of the Goldstone boson masses is $\delta = m^2 / (p_F \Delta)$. The first term in this expansion comes from the $O(M^2)$ term in (40), but the coefficients A contain the additional small parameter $\epsilon = (\Delta / p_F)$. In a combined expansion in δ and ϵ the $O(\epsilon\delta)$ mass term and the $O(\delta^2)$ chemical potential term appear at the same order. At this order, the masses of the flavored Goldstone bosons are

$$\begin{aligned} m_{\pi^\pm} &= \mp \frac{m_d^2 - m_u^2}{2p_F} + \left[\frac{4A}{f_\pi^2} (m_u + m_d) m_s \right]^{1/2}, \\ m_{K^\pm} &= \mp \frac{m_s^2 - m_u^2}{2p_F} + \left[\frac{4A}{f_\pi^2} m_d (m_u + m_s) \right]^{1/2}, \\ m_{K^0, \bar{K}^0} &= \mp \frac{m_s^2 - m_d^2}{2p_F} + \left[\frac{4A}{f_\pi^2} m_u (m_d + m_s) \right]^{1/2}. \end{aligned} \quad (43)$$

The masses of neutral mesons are unaffected by the effective chemical potential term. The elements of the mass matrix are

$$\begin{aligned}
m_{11}^2 &= \frac{8A}{3f_{\eta'}^2} [m_s(m_u + m_d) + m_u m_d], & m_{13}^2 &= -\frac{4\sqrt{2}A}{\sqrt{3}f_{\eta'}f_\pi} (m_u - m_d)m_s, \\
m_{33}^2 &= \frac{4A}{f_\pi^2} m_s(m_u + m_d), & m_{18}^2 &= \frac{8A}{3\sqrt{2}f_{\eta'}f_\pi} [m_s(m_u + m_d) - 2m_u m_d], \\
m_{88}^2 &= \frac{4A}{3f_\pi^2} [m_s(m_u + m_d) + 4m_u m_d], & m_{38}^2 &= -\frac{4A}{\sqrt{3}f_\pi^2} (m_u - m_d)m_s.
\end{aligned} \tag{44}$$

The flavor-octet and flavor-singlet decay constants f_π and $f_{\eta'}$ can be matched against the zero-momentum axial-vector current correlation functions in the high density effective theory. To leading order in g [19,24,36,37]

$$f_\pi^2 = \frac{21 - 8 \log(2)}{18} \left(\frac{p_F^2}{2\pi^2} \right), \quad f_{\eta'}^2 = \frac{3}{4} \left(\frac{p_F^2}{2\pi^2} \right). \tag{45}$$

We have identified the mass of a Goldstone mode with the energy of a $\vec{p} = 0$ excitation. We observe that the mass of the flavored Goldstone modes can become negative. Indeed, as observed in [15,17] this is likely to be the case for physically relevant values of the quark masses and the baryon density. This instability corresponds to the onset of meson condensation [15,17]. In the meson condensed phase the groundstate is reorganized and the masses are determined by small fluctuations around the new groundstate. This problem was recently studied in [38,39].

VII. SUMMARY

We have studied the effect of non-zero quark masses in effective theories of high density quark matter. We first studied the effective theory for quasi-particles in the vicinity of the Fermi surface. At leading order in $1/p_F$ there is a chirality conserving two fermion operator which arises from the mass correction to the kinetic energy of a quark in the vicinity of the Fermi surface. At next-to-leading order we find a chirality violating four-fermion operator which originates from mass corrections to the quark-quark scattering amplitude. We have argued that this scattering amplitude can be matched against a local operator in the effective theory.

In the effective chiral theory for Goldstone modes in the CFL phase the chirality conserving two-fermion operators correspond to an effective chemical potential. This effective chemical potential shifts the energy of flavored Goldstone modes. The chirality violating four-fermion operators correspond to meson mass terms. The coefficients of the meson mass terms are suppressed by Δ/p_F . We compute the Goldstone boson masses and find agreement with earlier results obtained by Son and Stephanov [19] and Bedaque and Schäfer [15].

We also studied the contribution of the anti-particle gap to the masses of Goldstone bosons. We show that to leading order in the coupling constant g the contribution from the anti-particle gap cancels against contributions from a mixed particle-anti-particle gap and the $O(M^2)$ shift in the particle gap. Individually, all these terms are non-vanishing, but they depend on the choice of gauge and therefore have no physical meaning.

Acknowledgements: We would like to thank S. Beane, P. Bedaque, T. Fugleberg, D. Son, and M. Stephanov for useful discussions. This work was supported in part by US DOE grant DE-FG-88ER40388.

REFERENCES

- [1] D. Bailin and A. Love, Phys. Rept. **107**, 325 (1984).
- [2] M. Alford, K. Rajagopal and F. Wilczek, Phys. Lett. **B422**, 247 (1998) [hep-ph/9711395].
- [3] R. Rapp, T. Schäfer, E. V. Shuryak and M. Velkovsky, Phys. Rev. Lett. **81**, 53 (1998) [hep-ph/9711396].
- [4] K. Rajagopal and F. Wilczek, hep-ph/0011333.
- [5] M. Alford, hep-ph/0102047.
- [6] T. Schäfer and E. Shuryak, nucl-th/0010049.
- [7] M. Alford, K. Rajagopal and F. Wilczek, Nucl. Phys. **B537**, 443 (1999) [hep-ph/9804403].
- [8] T. Schäfer and F. Wilczek, Phys. Rev. Lett. **82**, 3956 (1999) [hep-ph/9811473].
- [9] T. Schäfer, Nucl. Phys. **B575**, 269 (2000) [hep-ph/9909574].
- [10] N. Evans, J. Hormuzdiar, S. D. Hsu and M. Schwetz, Nucl. Phys. B **581**, 391 (2000) [hep-ph/9910313].
- [11] M. Alford, J. Berges and K. Rajagopal, Nucl. Phys. **B558**, 219 (1999) [hep-ph/9903502].
- [12] T. Schäfer and F. Wilczek, Phys. Rev. **D60**, 074014 (1999) [hep-ph/9903503].
- [13] M. Alford, J. Bowers and K. Rajagopal, Phys. Rev. D **63**, 074016 (2001) [hep-ph/0008208].
- [14] T. Schäfer, Phys. Rev. Lett. **85**, 5531 (2000) [nucl-th/0007021].
- [15] P. F. Bedaque and T. Schäfer, hep-ph/0105150.
- [16] M. Alford, K. Rajagopal, S. Reddy and F. Wilczek, hep-ph/0105009.

- [17] D. B. Kaplan and S. Reddy, hep-ph/0107265.
- [18] R. Casalbuoni and D. Gatto, Phys. Lett. **B464**, 111 (1999) [hep-ph/9908227].
- [19] D. T. Son and M. Stephanov, Phys. Rev. **D61**, 074012 (2000) [hep-ph/9910491], erratum: hep-ph/0004095.
- [20] M. Rho, A. Wirzba, and I. Zahed, Phys. Lett. **B473**, 126 (2000) [hep-ph/9910550].
- [21] D. K. Hong, T. Lee, and D. Min, Phys. Lett. **B477**, 137 (2000) [hep-ph/9912531].
- [22] C. Manuel and M. H. Tytgat, Phys. Lett. **B479**, 190 (2000) [hep-ph/0001095].
- [23] M. Rho, E. Shuryak, A. Wirzba and I. Zahed, Nucl. Phys. A **676**, 273 (2000) [hep-ph/0001104].
- [24] S. R. Beane, P. F. Bedaque, and M. J. Savage, Phys. Lett. **B483**, 131 (2000) [hep-ph/0002209].
- [25] D. K. Hong, Phys. Rev. D **62**, 091501 (2000) [hep-ph/0006105].
- [26] D. K. Hong, Phys. Lett. B **473**, 118 (2000) [hep-ph/9812510].
- [27] D. K. Hong, Nucl. Phys. B **582**, 451 (2000) [hep-ph/9905523].
- [28] R. Casalbuoni, R. Gatto and G. Nardulli, Phys. Lett. B **498**, 179 (2001) [hep-ph/0010321].
- [29] D. T. Son, Phys. Rev. D **59**, 094019 (1999) [hep-ph/9812287].
- [30] T. Schäfer and F. Wilczek, Phys. Rev. **D60**, 114033 (1999) [hep-ph/9906512].
- [31] D. K. Hong, V. A. Miransky, I. A. Shovkovy and L. C. Wijewardhana, Phys. Rev. **D61**, 056001 (2000) [hep-ph/9906478].
- [32] R. D. Pisarski and D. H. Rischke, Phys. Rev. **D61**, 074017 (2000) [nucl-th/9910056].
- [33] W. E. Brown, J. T. Liu and H. Ren, Phys. Rev. **D61**, 114012 (2000) [hep-ph/9908248].

- [34] T. Schäfer and F. Wilczek, Phys. Lett. B **450**, 325 (1999) [hep-ph/9810509].
- [35] R. D. Pisarski and D. H. Rischke, Phys. Rev. D **60**, 094013 (1999) [nucl-th/9903023].
- [36] K. Zarembo, Phys. Rev. D **62**, 054003 (2000) [hep-ph/0002123].
- [37] V. A. Miransky, I. A. Shovkovy and L. C. Wijewardhana, Phys. Rev. D **63**, 056005 (2001) [hep-ph/0009173].
- [38] V. A. Miransky and I. A. Shovkovy, hep-ph/0108178.
- [39] T. Schäfer, D. T. Son, M. A. Stephanov, D. Toublan and J. J. Verbaarschot, hep-ph/0108210.

FIGURES

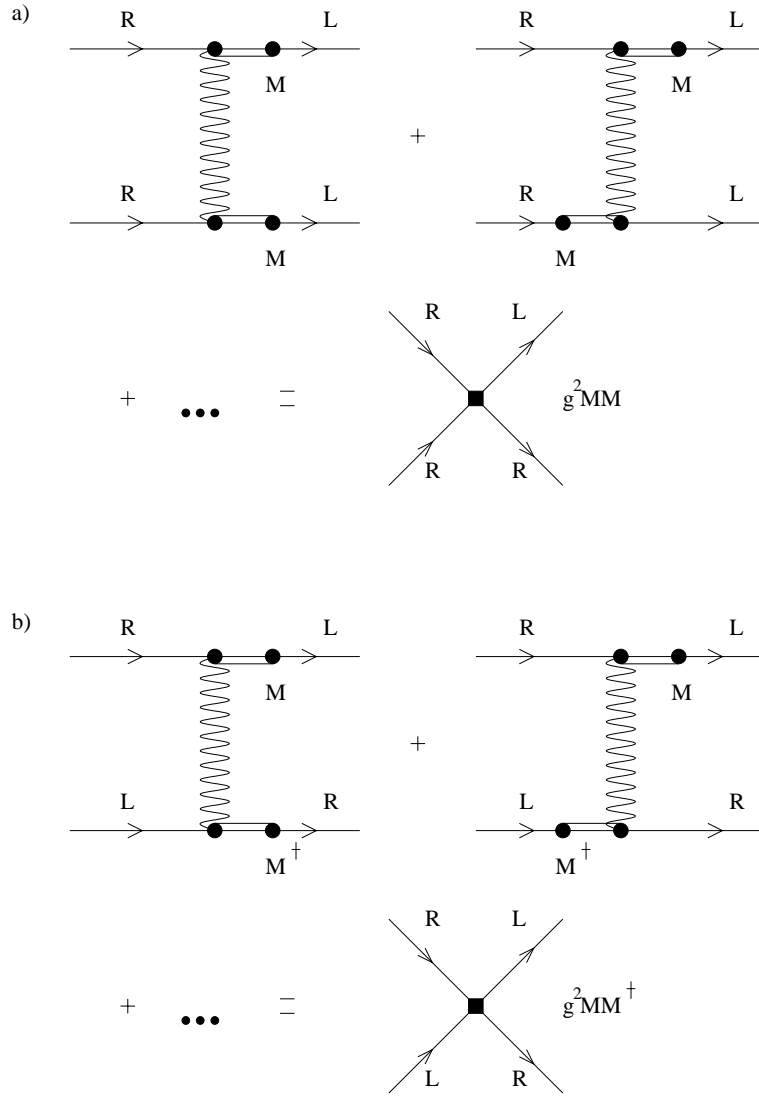


FIG. 1. This figure shows the effective chirality violating four-fermion vertices. Fig. a) shows the $RR \rightarrow LL$ vertex which is proportional to MM and Fig. b) shows the $RL \rightarrow LR$ vertex which is proportional to MM^\dagger . As explained in the text, there are also scattering amplitudes with two mass insertions on the same fermion line. These amplitudes are proportional to MM^\dagger , like the amplitudes in Fig. b, but they do not violate chirality, and they cannot be represented as local four-fermion operators.

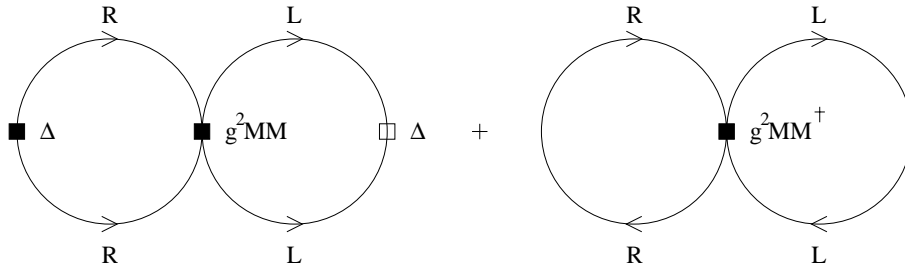


FIG. 2. This figure shows the contribution to the vacuum energy which arises from the four-fermion vertices shown in Fig. 1. Note that the second diagram does not contribute to the masses of Goldstone bosons.

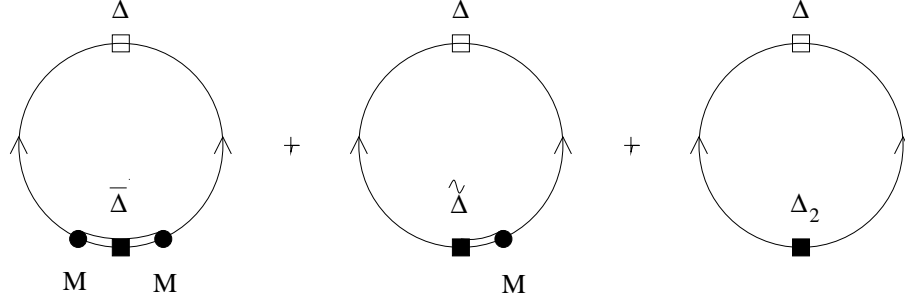


FIG. 3. This figure shows the contribution to the vacuum energy which arises from anti-particle gap, the mixed particle-anti-particle gap, and the $O(M^2)$ shift shift in the particle gap. As explained in the text, the sum of all the diagrams shown in this figure vanishes to leading order in g .

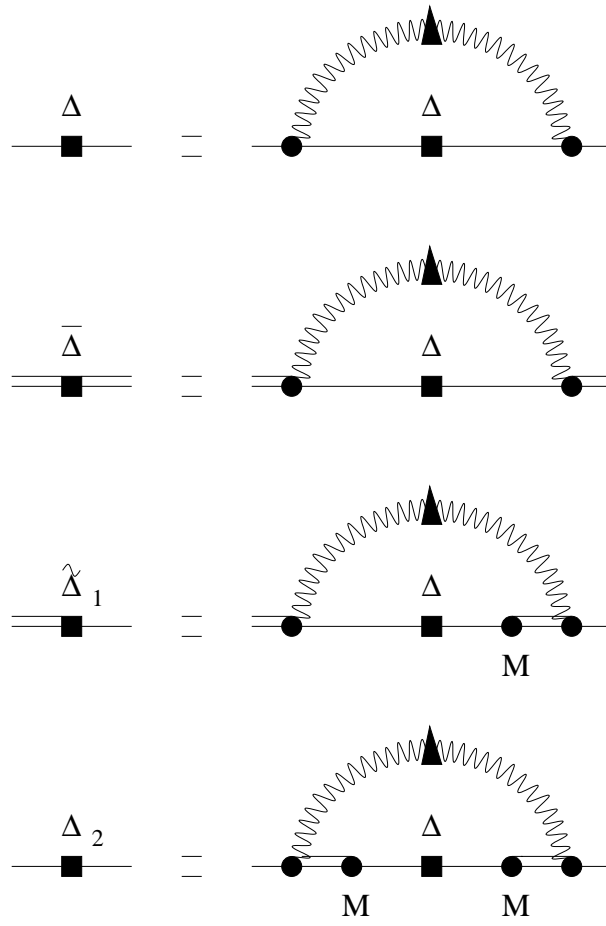


FIG. 4. This figure shows the gap equations for the particle gap Δ , the anti-particle gap $\bar{\Delta}$, the mixed particle-anti-particle gap $\tilde{\Delta}$, and the $O(M^2)$ shift in Δ . Solid lines denote ψ_+ propagators and double lines denote ψ_- propagators. The filled circles labeled M denote mass insertions, and the filled triangle denotes a gluon self energy insertion.